On Principal Angles Between Subspaces in $\mathbb{R}^n$*

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ABSTRACT

Let L, M be subspaces in $\mathbb{R}^n$, $\dim L = l \leqslant \dim M = m$. Then the *principal angles* between L and M, $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_l \leqslant \pi/2$, are given by

$$\cos\theta_i = \frac{\langle \mathbf{x}_i, \mathbf{y}_i \rangle}{\|\mathbf{x}_i\| \, \|\mathbf{y}_i\|}$$

$$= \max\left\{ \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|} : \begin{matrix} \mathbf{x} \in L, & \mathbf{x} \perp \mathbf{x}_k, \\ \mathbf{y} \in M, & \mathbf{y} \perp \mathbf{y}_k, \end{matrix} \; k = 1, \ldots, i-1 \right\}$$

where $(\mathbf{x}_i, \mathbf{y}_i) \in L \times M$, $i = 1, \ldots, l$, are the corresponding pairs of *principal vectors*. We also define $\sin\{L, M\} := \prod_{i=1}^{l} \sin\theta_i$, $\cos\{L, M\} := \prod_{i=1}^{l} \cos\theta_i$. We study relations between the principal angles and the *volume* of a matrix $A \in \mathbb{R}_r^{m \times n}$ defined by $\operatorname{vol} A := \sqrt{\sum \det^2 A_{IJ}}$, summing over all $r \times r$ submatrices A_{IJ} of A. Sample results are the following generalizations of the Hadamard and Cauchy-Schwarz inequalities:
1. Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}_l^{n \times n_1}$, $A_2 \in \mathbb{R}_m^{n \times n_2}$, rank $A = l + m$; then $\operatorname{vol} A = \operatorname{vol} A_1 \operatorname{vol} A_2 \sin\{R(A_1), R(A_2)\}$.
2. Let $B, C \in \mathbb{R}_r^{n \times r}$; then

$$|\det(B^T C)| = \operatorname{vol} B \operatorname{vol} C \cos\{R(B), R(C)\}.$$

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1. INTRODUCTION

Let L, M be subspaces in $\mathbb{R}^n$, and $\dim L = l \leqslant \dim M = m$. Then the *principal angles* between L and M,

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_l \leqslant \frac{\pi}{2}, \tag{1.1}$$

are defined by [1]

$$\begin{aligned}\cos\theta_i &:= \frac{\langle \mathbf{x}_i, \mathbf{y}_i\rangle}{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|} \\ &= \max\left\{ \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|} : \begin{array}{ll} \mathbf{x} \in L, & \mathbf{x} \perp \mathbf{x}_k, \\ \mathbf{y} \in M, & \mathbf{y} \perp \mathbf{y}_k, \end{array} \; k = 1, \ldots, i-1 \right\},\end{aligned} \tag{1.2}$$

where

$$(\mathbf{x}_i, \mathbf{y}_i) \in L \times M, \qquad i = 1, \ldots, l, \tag{1.3}$$

are the corresponding l pairs of *principal vectors*. Note that

$$\theta_1 = \cdots = \theta_k = 0 < \theta_{k+1} \quad \text{iff} \quad \dim L \cap M = k, \tag{1.4}$$

and that if $\dim L = \dim M = 1$, then θ_1 is the (nonobtuse) angle between the lines L and M.

We also denote the product of principal sines, and the product of principal cosines, by

$$\sin\{L, M\} := \sin\theta_1 \cdots \sin\theta_l, \tag{1.5}$$

$$\cos\{L, M\} := \cos\theta_1 \cdots \cos\theta_l. \tag{1.6}$$

Note that (1.5) and (1.6) are just notation, and not ordinary trigonometrical functions. In particular, $\sin^2\{L, M\} + \cos^2\{L, M\} \leqslant 1$.

Principal angles were introduced by Afriat in his study [1] of the geometry of subspaces in $\mathbb{R}^n$ in terms of their orthogonal and oblique projectors. An important application of principal angles in statistics is the canonical correlation theory of Hotelling [11]; see also [5].

Principal angles and vectors generalize least squares solutions in the following sense: If $\dim L = 1$, say L is the line spanned by the vector $\mathbf{a}$, then the principal angle and vector between L and M are found by minimizing $\{\|\mathbf{a}-\mathbf{y}\|_2 : \mathbf{y} \in M\}$.

Björck and Golub [3] used the singular value decomposition to compute the principal angles as follows:

LEMMA 1. *Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and let*

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_l \geqslant 0 \tag{1.7}$$

be the singular values of $Q_M^T Q_L$. Then

$$\cos\theta_i = \sigma_i, \qquad i = 1,\dots,l, \tag{1.8}$$

and

$$\sigma_1 = \cdots = \sigma_k = 1 > \sigma_{k+1} \quad \textit{iff} \quad \dim L \cap M = k. \tag{1.9}$$

In this paper, we discuss some relations between principal angles and the matrix volume defined in [2]. In Section 3, we express the principal cosines and principal sines in terms of the volume function. Principal angles allow us to "equalize" the Hadamard and Cauchy-Schwarz inequalities in Section 4.

2. PRELIMINARY RESULTS

Let $\mathbb{R}_r^{m\times n}$ be the set of $m\times n$ matrices of rank r. The *k-dimensional volume* of $A \in \mathbb{R}_r^{m\times n}$, $0 < k \leqslant r$, is defined as

$$\operatorname{vol}_k A := \prod_{i=1}^{k} \sigma_i, \tag{2.1}$$

where

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0 \tag{2.2}$$

are the nonzero *singular values* of A. In particular, the *r-dimensional volume* of $A \in \mathbb{R}_r^{m \times n}$ is called its *volume* and denoted by vol A [2]:

$$\operatorname{vol} A := \begin{cases} 0 & \text{if} \quad r = 0, \\ \prod_{i=1}^{r} \sigma_i & \text{if} \quad r > 0, \end{cases} \tag{2.3}$$

or equivalently

$$\operatorname{vol} A = \sqrt{\Sigma \det^2 A_{IJ}}, \tag{2.4}$$

summing over all $r \times r$ submatrices A_{IJ} of A. For consistency, set

$$\operatorname{vol}_0 A := \min\{1, \operatorname{rank} A\}. \tag{2.5}$$

In particular, if A has full column rank, then

$$\operatorname{vol} A = \sqrt{\det(A^T A)}, \tag{2.6}$$

the volume of the parallelepiped spanned by the columns of A.

The volume function is closely related to compound matrices. The kth *compound matrix* of A, $C_k(A)$, is the $\binom{m}{k} \times \binom{n}{k}$ matrix whose elements are determinants of all $k \times k$ submatrices of A in lexicographic order. The singular values of $C_k(A)$ are all products $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k}$ of singular values of A. Therefore the largest singular value of $C_k(A)$ (its *spectral norm*) is $\operatorname{vol}_k A$. Some well-known properties of compound matrices are collected below (see e.g. [8]).

PROPOSITION 1.

(a) *If* $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, $\operatorname{rank} AB = r$, *then* $C_k(AB) = C_k(A) C_k(B)$, $k \leqslant \min\{m, r, n\}$.

(b) $C_k(A^T) = C_k(A)^T$.

(c) $C_k(I) = I$ (*appropriate size identity matrix*).

(d) *If A has orthonormal columns, so does* $C_k(A)$.

(e) $\operatorname{vol}_k A = \|C_k(A)\|_2$.

3. PRINCIPAL ANGLES AND VOLUME

The following lemma is used in the sequel.

LEMMA 2. *Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and denote*

$$Q := (Q_M, Q_L). \tag{3.1}$$

Let the singular values of $Q_M^T Q_L$ be

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_l \geqslant 0. \tag{3.2}$$

Then the singular values of Q, in decreasing order, are

$$\sqrt{1+\sigma_1}, \dots, \sqrt{1+\sigma_l}, \underbrace{1,\dots,1}_{m-l}, \sqrt{1-\sigma_l}, \dots, \sqrt{1-\sigma_1}. \tag{3.3}$$

Proof. Since

$$Q^TQ = \begin{pmatrix} I & Q_M^TQ_L \\ Q_L^TQ_M & I \end{pmatrix} = I + \begin{pmatrix} O & Q_M^TQ_L \\ Q_L^TQ_M & O \end{pmatrix},$$

the eigenvalues of

$$\begin{pmatrix} O & Q_M^TQ_L \\ Q_L^TQ_M & O \end{pmatrix}$$

are (see [9, p. 418])

$$\pm\sigma_1, \pm\sigma_2, \dots, \pm\sigma_l, \underbrace{0,\dots,0}_{m-l}.$$

Thus the eigenvalues of Q^TQ are

$$1\pm\sigma_1, \dots, 1\pm\sigma_l, \underbrace{1,\dots,1}_{m-l}. \qquad \blacksquare$$

THEOREM 1. *Let the columns of $Q_L \in \mathbb{R}^{n\times l}$ and $Q_M \in \mathbb{R}^{n\times m}$ be orthonormal bases for L and M respectively, and denote*

$$Q := (Q_M, Q_L). \tag{3.4}$$

Let

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_l \leqslant \frac{\pi}{2} \tag{3.5}$$

be the principal angles between L and M. Then

$$\cos\theta_{l-i} = 1 - \frac{\mathrm{vol}^2_{m+i+1} Q}{\mathrm{vol}^2_{m+i} Q} = \frac{\mathrm{vol}^2_{l-i} Q}{\mathrm{vol}^2_{l-i-1} Q} - 1, \qquad i = 0, 1, \ldots, l-k-1, \tag{3.6}$$

$$\sin\theta_{l-i} = \frac{\mathrm{vol}_{m+l+1} Q}{\mathrm{vol}_{l-i-1} Q} (\sin\theta_{l-i+1} \cdots \sin\theta_l)^{-1}, \qquad i = 0, 1, \ldots, l-k-1, \tag{3.7}$$

where

$$k = \dim L \cap M. \tag{3.8}$$

Proof. By Lemmas 1 and 2,

$$\mathrm{vol}_{m+i+1} Q = (\mathrm{vol}_{m+i} Q)\sqrt{1 - \cos\theta_{l-i}}, \qquad i = 0, 1, \ldots, l-1,$$

$$\mathrm{vol}_{l-i} Q = (\mathrm{vol}_{l-i-1} Q)\sqrt{1 + \cos\theta_{l-i}}, \qquad i = 0, 1, \ldots, l-1,$$

$$\mathrm{vol}_{m+i+1} Q = \mathrm{vol}_{l-i-1} Q \sin\theta_{l-i} \cdots \sin\theta_l, \qquad i = 0, 1, \ldots, l-1,$$

which, after some arithmetic calculations, prove (3.6) and (3.7). ∎

The following theorem gives the analogous results in terms of the orthogonal projectors P_L and P_M on L and M respectively.

THEOREM 2. *Let P_L and P_M be the orthogonal projectors on L and M respectively,*

$$P := (P_M, P_L), \tag{3.9}$$

and let

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_l \leqslant \frac{\pi}{2} \tag{3.10}$$

be the principal angles between L and M. Then

$$\cos\theta_{l-i} = 1 - \frac{\operatorname{vol}^2_{m+i+1} P}{\operatorname{vol}^2_{m+i} P} = \frac{\operatorname{vol}^2_{l-i} P}{\operatorname{vol}^2_{l-i-1} P} - 1, \qquad i = 0, 1, \ldots, l-k-1, \tag{3.11}$$

$$\sin\theta_{l-i} = \frac{\operatorname{vol}_{m+l+1} P}{\operatorname{vol}_{l-i-1} P} (\sin\theta_{l-i+1} \cdots \sin\theta_l)^{-1}, \qquad i = 0, 1, \ldots, l-k-1, \tag{3.12}$$

where

$$k = \dim L \cap M. \tag{3.13}$$

Proof. Let Q_L, Q_M be as in Lemma 2. Then

$$P_L = Q_L Q_L^T, \qquad P_M = Q_M Q_M^T.$$

Therefore

$$P = (Q_M, Q_L) \begin{pmatrix} Q_M^T & O \\ O & Q_L^T \end{pmatrix} = QU^T,$$

where

$$Q = (Q_M, Q_L)$$

and

$$U=\begin{pmatrix} Q_M & O \\ O & Q_L \end{pmatrix}.$$

Thus U has orthonormal columns, and

$$\begin{aligned}\operatorname{vol}_i P &= \|C_i(Q)\|_2 \qquad \text{(by Proposition 1)}\\ &= \operatorname{vol}_i Q, \qquad i=0,1,\dots,m+l.\end{aligned}$$

The result follows from Theorem 1. ■

In applications to linear equations the subspaces are the null spaces of the coefficient matrices. Let

$$A_i\mathbf{x}=\mathbf{b}_i, \qquad i=1,2, \tag{3.14}$$

be two linear systems, with nonempty solution sets

$$S_i=\{\mathbf{x}: A_i\mathbf{x}=\mathbf{b}_i\}, \qquad i=1,2. \tag{3.15}$$

Then the principal angles between S_1 and S_2 are the principal angles between $L=N(A_1)$ and $M=N(A_2)$. The respective projectors are

$$P_L=I-A_1^\dagger A_1, \qquad P_M=I-A_2^\dagger A_2. \tag{3.16}$$

The following theorem suggests that instead of using P_L and P_M, we can use

$$P_{L^\perp}=A_1^\dagger A_1, \qquad P_{M^\perp}=A_2^\dagger A_2 \tag{3.17}$$

to compute the principal angles between L, M.

THEOREM 3. *The nonzero principal angles between L, M are equal to the nonzero principal angles between $L^\perp, M^\perp$.*

Proof. Let $(\mathbf{x}_i,\mathbf{y}_i)$ be a pair of principal vectors corresponding to the ith nonzero principal angle θ_i between L and M. Then $P_L\mathbf{y}_i=\alpha\mathbf{x}_i$ for some scalar α. Let the two vectors $\mathbf{x}_i^\perp$ and $\mathbf{y}_i^\perp$ be obtained from $\mathbf{x}_i$ and $\mathbf{y}_i$ respectively, by a $\pi/2$ rotation in $\{\mathbf{x}_i,\mathbf{y}_i\}$ plane. Then $\mathbf{x}_i^\perp\in L^\perp$, $\mathbf{y}_i^\perp\in M^\perp$, and the angle between $x_i^{\perp}$ and $y_i^{\perp}$ is also θ_i. Therefore the ith nonzero principal angle between $L^{\perp}, M^{\perp}$ is $\leqslant \theta_i$. The result follows by interchanging L, M with $L^{\perp}, M^{\perp}$. ∎

4. HADAMARD AND CAUCHY-SCHWARZ EQUALITIES

Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be a basis for M, and let L be 1-dimensional, say $L = \operatorname{span}\{\mathbf{a}\}$, where $\mathbf{a} = \mathbf{a}_M + \mathbf{a}_{M^{\perp}}$, $\mathbf{a}_M \in M$, and $\mathbf{a}_{M^{\perp}} \in M^{\perp}$. Then [8]

$$\|\mathbf{a}_{M^{\perp}}\|_2 = \frac{\operatorname{vol}_{m+1}(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{a})}{\operatorname{vol}(\mathbf{a}_1, \dots, \mathbf{a}_m)}. \tag{4.1}$$

That is,

$$\operatorname{vol}_{m+1}(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{a}) = \operatorname{vol}(\mathbf{a}_1, \dots, \mathbf{a}_m)\|\mathbf{a}\|_2 \sin\theta, \tag{4.2}$$

where θ is the principal angle between $\mathbf{a}$ and M. For the general case, Afriat [1] gave the following "equalized" Hadamard inequality.

Lemma 3. *Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}_l^{n\times l}$, $A_2 \in \mathbb{R}_m^{n\times m}$. Then*

$$\operatorname{vol}_{l+m} A = \operatorname{vol} A_1 \operatorname{vol} A_2 \sin\{R(A_1), R(A_2)\}, \tag{4.3}$$

where $\sin\{R(A_1), R(A_2)\}$ is the product of principal sines between $R(A_1)$ and $R(A_2)$: see (1.5).

In Lemma 3 the matrices A_1, A_2 are of full column rank. A further generalization of the Hadamard inequality is:

Theorem 4. *Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}_l^{n\times n_1}$, $A_2 \in \mathbb{R}_m^{n\times n_2}$, $\operatorname{rank} A = l + m$. Then*

$$\operatorname{vol} A = \operatorname{vol} A_1 \operatorname{vol} A_2 \sin\{R(A_1), R(A_2)\}. \tag{4.4}$$

Proof.

$$\operatorname{vol}^2 A = \sum_J \operatorname{vol}^2 A_{*J},$$

where the summation is over all $n \times (l+m)$ submatrices of rank $l+m$. Since every $n \times (l+m)$ submatrix of rank $l+m$ has l columns A_{*J_1} from A_1 and m columns A_{*J_2} from A_2, then

$$\begin{aligned}\operatorname{vol}^2 A &= \sum_{J_1}\sum_{J_2} \operatorname{vol}^2(A_{*J_1}, A_{*J_2}),\\ &= \sum_{J_1}\sum_{J_2} \operatorname{vol}^2 A_{*J_1} \operatorname{vol}^2 A_{*J_2} \sin^2\{R(A_1), R(A_2)\} \qquad \text{(by Lemma 3)}\\ &= \operatorname{vol}^2 A_1 \operatorname{vol}^2 A_2 \sin^2\{R(A_1), R(A_2)\}. \qquad \blacksquare\end{aligned}$$

For a square matrix A, the above results imply:

COROLLARY 1. *Let $A = (A_1, A_2)$ be a square matrix, and $A_1 \in \mathbb{R}^{n\times l}$, $A_2 \in \mathbb{R}^{n\times m}$. Then*

$$|\det A| = \operatorname{vol}_l A_1 \operatorname{vol}_m A_2 \sin\{R(A_1), R(A_2)\}. \tag{4.5}$$

Now we "equalize" the Cauchy-Schwarz inequality.

THEOREM 5. *Let $B, C \in \mathbb{R}_r^{n\times r}$. Then*

$$|\det(B^T C)| = \operatorname{vol} B \operatorname{vol} C \cos\{R(B), R(C)\}, \tag{4.6}$$

where $\cos\{R(B), R(C)\}$ *is the product of principal cosines between* $R(B)$ *and* $R(C)$: *see* (1.6).

Proof. Let Q_B and Q_C be orthonormal bases for $R(B)$ and $R(C)$, respectively, so that,

$$B = Q_B R_B, \qquad C = Q_C R_C$$

for some matrices $R_B, R_C \in \mathbb{R}_r^{r\times r}$. Then

$$\begin{aligned}|\det(B^T C)| &= |\det R_B^T|\,|\det R_C|\,|\det(Q_B^T Q_C)|,\\ &= \operatorname{vol} B \operatorname{vol} C \cos\{R(B), R(C)\}, \qquad \text{by Lemma 1.}\end{aligned}$$

This completes the proof. $\blacksquare$

Denote the set of strictly increasing sequences of k elements from $1,\dots,n$ by

$$Q_{k,n} := \{I = (i_1,\dots,i_k) : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

For any sets $I \subset \{1,2,\dots,m\}$, $J \subset \{1,2,\dots,n\}$, let A_{I*}, A_{*J}, A_{IJ} denote the submatrices of A lying in rows indexed by I, in columns indexed by J, and in their intersection, respectively. For $A \in \mathbb{R}_r^{m\times n}$, let

$$\mathscr{I}(A) := \{I \in Q_{r,m} : \operatorname{rank} A_{I*} = r\}, \qquad \mathscr{J}(A) := \{J \in Q_{r,n} : \operatorname{rank} A_{*J} = r\}.$$

For $I \subset \{1,2,\dots,m\}$ define the subspace $\mathbb{R}_I^m$ by

$$\mathbb{R}_I^m := \{x = (x_1,\dots,x_m) \in \mathbb{R}^m : x_i = 0 \text{ if } i \notin I\}.$$

An interesting interpretation of the determinants of maximal nonsingular submatrices is the following

Corollary 2. *Let $A \in \mathbb{R}_r^{m\times n}$, $I \in Q_{r,m}$. Then*

$$\cos\{R(A), \mathbb{R}_I^m\} = \frac{|\det A_{IJ}|}{\operatorname{vol} A_{*J}} \tag{4.7}$$

for any $J \in \mathscr{J}(A)$.

Proof. Let

$$I = \{i_1, i_2, \dots, i_r\}, \qquad B = (e_{i_1},\dots,e_{i_r}), \quad \text{and} \quad C = A_{*J} \quad \forall J \in \mathscr{J}(A).$$

Then

$$R(B) = \mathbb{R}_I^m, \qquad R(C) = R(A), \quad \text{and} \quad B^T C = A_{IJ}.$$

By Theorem 5

$$\cos\{R(A), \mathbb{R}_I^m\} = \cos\{R(B), R(C)\} = \frac{|\det A_{IJ}|}{\operatorname{vol} A_{*J}}. \qquad \blacksquare$$

Note that for any $I \in Q_{r,m}$, the ratio $|\det A_{IJ}|/\mathrm{vol}\, A_{*J}$ is independent of the choice of $J \in \mathscr{J}(A)$.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \tag{4.8}$$

$I = \{1,2\}$, $J = \{1,2\}$. Then A is of rank 2, and

$$|\det A_{IJ}| = 3, \qquad \mathrm{vol}\, A_{*J} = \sqrt{3^2 + 6^2 + 3^2} = 3\sqrt{6},$$

and, by (4.7),

$$\cos\left\{R(A), \mathbb{R}^3_{\{1,2\}}\right\} = \frac{1}{\sqrt{6}}.$$

COROLLARY 3. *Let $L \subset \mathbb{R}^m$ be a subspace of dimension r. Then*

$$\sum_{I \in Q_{r,m}} \cos^2\{L, \mathbb{R}^m_I\} = 1. \tag{4.9}$$

Proof. Follows from (4.7), since

$$\mathrm{vol}^2 A_{*J} = \sum_{I \in \mathscr{I}(A)} \det{}^2 A_{IJ}. \qquad \blacksquare$$

The corresponding result for the complementary orthogonal subspace $L^\perp$ is

$$\sum_{I \in Q_{r,m}} \cos^2\left\{L^\perp, \mathbb{R}^m_{I^c}\right\} = 1, \tag{4.10}$$

where I^c is the *complement* of I in $\{1,2,\ldots,m\}$. The equivalence of (4.9) and (4.10) is by

$$\cos\{L, \mathbb{R}^m_I\} = \cos\left\{L^\perp, \mathbb{R}^m_{I^c}\right\}, \tag{4.11}$$

see Theorem 3.

EXAMPLE 2. Let $L = R(A)$ for A of (4.8). Then $L^{\perp} = N(A^T)$ is the line spanned by $(1, -2, 1)$. The complement of $I = \{1,2\}$ is $I^c = \{3\}$, and $\mathbb{R}^3_{I^c}$ is the x_3-axis. The angle between the lines $L^{\perp}$ and $\mathbb{R}^3_{I^c}$, $\angle(L^{\perp}, \mathbb{R}^3_{I^c})$, is given by

$$\cos\angle\left(L^{\perp}, \mathbb{R}^3_{I^c}\right) = \cos\angle((1,-2,1),(0,0,1)) = \frac{1}{\sqrt{1^2+2^2+1^2}} = \frac{1}{\sqrt{6}},$$

in agreement with (4.11) and Example 1.

THEOREM 6. *Let* $B \in \mathbb{R}^{n\times r}_r$, $C \in \mathbb{R}^{n\times s}$, *and* $\operatorname{rank} C \geqslant r$. *Then*

$$\cos\{R(B), R(C)\} = \cos\angle\{C_r(B), R(C_r(C))\}, \tag{4.12}$$

where $\angle\{C_r(B), R(C_r(C))\}$ *is the angle between the vector* $C_r(B)$ *and the subspace* $R(C_r(C))$.

Proof. Let the columns of

$$E = (e_1, \dots, e_r) \quad \text{and} \quad F = (f_1, \dots, f_t)$$

be orthonormal bases for $R(B)$ and $R(C)$ respectively, and let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r \geqslant 0$ be the singular values of F^TE. Then by Lemma 1

$$\cos\{R(B), R(C)\} = \sigma_1 \cdots \sigma_r.$$

On the other hand, the columns of $C_r(E)$ and $C_r(F)$ form orthonormal bases for $C_r(B)$ and $R(C_r(C))$ respectively. Moreover, the singular value of $C_r(F)^TC_r(E)$ $[= C_r(F^TE)]$ is

$$\sigma_1 \cdots \sigma_r.$$

By the same lemma,

$$\cos\angle\{C_r(B), R(C_r(C))\} = \sigma_1 \cdots \sigma_r. \qquad \blacksquare$$

We now use the above results (the equalized Hadamard and Cauchy-Schwarz inequalities) to equalize a determinantal inequality of Thompson.

THEOREM 7.[1] *Let* $A \in \mathbb{R}^{kr \times kr}$ *be partitioned as* $A = (B_1, B_2, \ldots, B_k)$, *where* $B_i \in \mathbb{R}_r^{kr \times r}$. *Then*

$$\det\begin{pmatrix} B_1^T B_1 & \cdots & B_1^T B_k \\ \vdots & \ddots & \vdots \\ B_k^T B_1 & \cdots & B_k^T B_k \end{pmatrix}$$

$$= \det\begin{pmatrix} \det(B_1^T B_1) & \cdots & \det(B_1^T B_k) \\ \vdots & \ddots & \vdots \\ \det(B_k^T B_1) & \cdots & \det(B_k^T B_k) \end{pmatrix}$$

$$\times \prod_{i=1}^{k-1} \frac{\sin^2\{R(B_i), R(B_{i+1}, \ldots, B_k)\}}{1 - \cos^2 \angle\{C_r(B_i), R(C_r(B_{i+1}), \ldots, C_r(B_k))\}}. \quad (4.13)$$

Proof. We prove the case $k = 2$:

$$\det\begin{pmatrix} B_1^T B_1 & B_1^T B_2 \\ B_2^T B_1 & B_2^T B_2 \end{pmatrix} = \det A^T A$$

$$= \mathrm{vol}_r^2 B_1 \,\mathrm{vol}_r^2 B_2 \sin^2\{R(B_1), R(B_2)\}, \qquad \text{by Corollary 1.}$$

Now let

$$\tilde{A} = (C_r(B_1), C_r(B_2)).$$

Then

$$\det\begin{pmatrix} \det(B_1^T B_1) & \det(B_1^T B_2) \\ \det(B_2^T B_1) & \det(B_2^T B_2) \end{pmatrix}$$

$$= \det \tilde{A}^T \tilde{A},$$

$$= \| C_r(B_1) \|_2^2 \| C_r(B_2) \|_2^2 \sin^2 \angle\{C_r(B_1), C_r(B_2)\},$$

$$= \mathrm{vol}_r^2 B_1 \,\mathrm{vol}_r^2 B_2 \left(1 - \cos^2 \angle\{C_r(B_1), C_r(B_2)\}\right).$$

The general case $k \geqslant 2$ is proved analogously. ■

[1] We proved this originally for $k = 2$, and we thank the referee for suggesting extension to $k > 2$.

REMARK. If the denominator in (4.13) is zero, then the numerator is also zero. In this case, (4.13) is also valid if we let the ratio be 1.

Note that $R(C_r(B_{i+1}),\dots,C_r(B_k)) \subseteq R(C_r(B_{i+1},\dots,B_k))$; then

$$\begin{aligned}
&\cos\angle\{C_r(B_i), R(C_r(B_{i+1}),\dots,C_r(B_k))\}\\
&\quad\leqslant \cos\angle\{C_r(B_i), R(C_r(B_{i+1},\dots,B_k))\},\\
&\quad= \cos\{R(B_i), R(B_{i+1},\dots,B_k)\}. \qquad (4.14)
\end{aligned}$$

The last equality follows from Theorem 6. If $\theta_1,\dots,\theta_r$ are the principal angles between two subspaces L and M, then we always have

$$\frac{\sin^2\{L,M\}}{1-\cos^2\{L,M\}} = \frac{\prod_{i=1}^{r}(1-\cos^2\theta_i)}{1-\prod_{i=1}^{r}\cos^2\theta_i} \leqslant 1. \qquad (4.15)$$

Thus from (4.14) and (4.15), we have

$$\begin{aligned}
&\prod_{i=1}^{k-1}\frac{\sin^2\{R(B_i), R(B_{i+1},\dots,B_k)\}}{1-\cos^2\angle\{C_r(B_i), R(C_r(B_{i+1}),\dots,C_r(B_k))\}}\\
&\quad\leqslant \prod_{i=1}^{k-1}\frac{\sin^2\{R(B_i), R(B_{i+1},\dots,B_k)\}}{1-\cos^2\{R(B_i), R(B_{i+1},\dots,B_k)\}} \leqslant 1. \qquad (4.16)
\end{aligned}$$

Therefore Theorem 7 implies the following inequalities of Davis [6], Everitt [7] ($k=2$), and Thompson [13] ($k \geqslant 2$).

COROLLARY 4. *Let a p.s.d. matrix A of order kr be partitioned as*

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k}\\ \vdots & \ddots & \vdots\\ A_{k1} & \cdots & A_{kk}\end{pmatrix}, \qquad \textit{where} \quad A_{ij} \in \mathbb{R}^{r\times r}. \qquad (4.17)$$

Then

$$\det\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} \leqslant \det\begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & \ddots & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{pmatrix}. \quad \blacksquare \tag{4.18}$$

A matrix $A \in \mathbb{R}_r^{n \times n}$ is called *range-Hermitian* (*or an EP matrix, or an* EP_r *matrix*) if

$$R(A^T) = R(A). \tag{4.19}$$

Note that the class of range-Hermitian matrices includes normal matrices and nonsingular matrices. A good reference on range-Hermitian matrices is [12].

A characterization of range-Hermitian matrices follows.

THEOREM 8. *Let* $A \in \mathbb{R}_r^{n \times n}$. *Then* A *is range-Hermitian if and only if* A *has* r *nonzero eigenvalues* $\lambda_1, \ldots, \lambda_r$ *and* $\operatorname{vol} A = \prod_{i=1}^{r} |\lambda_i|$.

Proof. Let $A = CB^T$ be a full-rank factorization of A. Since

$$R(A^T) = R(B), \qquad R(A) = R(C),$$

we have

$$R(A^T) = R(A) \quad \text{if and only if} \quad \cos\{R(B), R(C)\} = 1.$$

By Theorem 6

$$\cos\angle(C_r(B), C_r(C)) = \frac{|C_r(B)^T C_r(C)|}{\|C_r(B)\|_2 \|C_r(C)\|_2} = \cos\{R(B), R(C)\} = 1.$$

And using the fact [2]

$$\operatorname{vol} A = \operatorname{vol} B \operatorname{vol} C,$$

then

$$\frac{|\Sigma \det A_{II}|}{\operatorname{vol} A} = 1,$$

where the summation is over all $r \times r$ principal submatrices of A. Finally note that the rth elementary symmetric function of the eigenvalues of A is the sum of all the $r \times r$ principal minors of A. ∎

EXAMPLE 3. The matrix A of (4.8) is range-Hermitian. Its nonzero eigenvalues are

$$\lambda_1 = \frac{15 + 3\sqrt{33}}{2}, \qquad \lambda_2 = \frac{15 - 3\sqrt{33}}{2},$$

and its nonzero singular values are, correct to six decimals,

$$\sigma_1 = 16.848103, \qquad \sigma_2 = 1.068369.$$

Therefore $\operatorname{vol} A = \sigma_1\sigma_2 = |\lambda_1||\lambda_2| = 18$.

To put Theorem 8 in perspective, recall that for any square matrix $A \in \mathbb{R}^{n \times n}$ with singular values $\sigma_1 \geqslant \cdots \geqslant \sigma_n \geqslant 0$ and eigenvalues $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$,

$$\prod_{i=1}^{k} |\lambda_i| \leqslant \prod_{i=1}^{k} \sigma_i, \qquad k = 1, \ldots, n, \tag{4.20}$$

with equality for $k = n$ [10, Theorem 3.3.2]. For $k = n$ the common value in (4.20) is nonzero iff A is nonsingular, in which case it is range-Hermitian. Therefore, the equality in (4.20) for $k = r = n$ is a special case of Theorem 8 which states that, for range-Hermitian matrices of rank r,

$$\prod_{i=1}^{r} |\lambda_i| = \prod_{i=1}^{r} \sigma_i. \tag{4.21}$$

Conversely, Theorem 8 follows from the nonsingular case, since a matrix A is range-Hermitian iff

$$A = U\begin{pmatrix} B & O \\ O & O \end{pmatrix}U^T, \qquad U \text{ orthogonal}, \quad B \text{ nonsingular};$$

see [4, Theorem 4.3.1]. Written analogously to (4.20), a characterization of normal matrices is [10, Problem 3.3.14]

$$\prod_{i=1}^{k} |\lambda_i| = \prod_{i=1}^{k} \sigma_i, \qquad k = 1,\dots,n. \tag{4.22}$$

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